On System of triple Diophantine Equations

$x+y=a^2\ ,\ 2x+y=-a^2+b^2\ ,\ x+2y=a^2+2c^2$

* Dr.V.Krithika[1], Dr.A.Vijayasankar[2], Dr.M.A.Gopalan[3]

[1] *Assistant Professor, Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli,*

[2] *Assistant Professor, Department of Mathematics, National College, Tiruchirappalli,*

[3] *Professor, Department of Mathematics, Shrimati Indira Gandhi College, Tiruchirappalli,*

[1] krithijay.24@gmail.com, [2] avsankar70@yahoo.com, [3] mayilgopalan@gmail.com

Abstract: *An attempt is made to obtain non-zero distinct integer quintuples* (x,y,a,b,c) *satisfying the system of three equations* $x+y=a^2, 2x+y=a^2+3b^2, x+2y=a^2+c^2$. *Different sets of integer solutions are presented.*

Keywords: System of triple equations, triple equations with five unknowns, Integer solutions.

1. INTRODUCTION:

In [1], an attempt has been made to obtain pairs of non-zero distinct integers x, y such that, in each pair

(i) $x+y=a^2, 2x+y=b^2, x+2y=a^3$

(ii) $x+y=a^2, 2x+y=b^2, x+2y=c^3$

In [2], Illustrates the analysis of obtaining different sets of distinct integer solutions to two systems of triple equations with five unknowns given by

(i) $x+y=a^2, 2x+y=b^2, x+2y=3c^2$

(ii) $x+y=a^2, 2x+y=b^2, x+2y=2c^2$ respectively.

In [3], the system of Triple Equations with five variables $x+y=a^2, 2x+y=b^2, x+2y=a^2-c^2$ is studied for its integer solutions. In [4] and [5], two more triple equations are analyzed for their integer solutions respectively.

This communication exhibits different sets of non-zero distinct integer solutions for the system of Triple equations with five unknowns given by $x+y=a^2\ ,\ 2x+y=-a^2+b^2\ ,\ x+2y=a^2+2c^2$.

2. NOTATIONS:

1. Regular Polygonal Number of rank n with sides m: $t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$

2. Pyramidal Number of rank n with sides m: $p_n^m = \frac{1}{6}[n(n+1)][(m-2)n+(5-m)]$

3. Gnomonic Number of rank n: $gn_n = 2n+1$

4. Stella Octangular Number of rank n: $SO_n = n(2n^2-1)$

5. Jacobsthal Number of rank n: $J_n = \frac{1}{3}\left[2^n - (-1)^n\right]$

6. Octahedral Number of rank n: $OH_n = \frac{1}{3}n(2n^2+1)$

7. Pentatope Number of rank n : $pt_n = \frac{n(n+1)(n+2)(n+3)}{24}$

3. METHOD OF ANALYSIS:

Consider the system of equations

$$x+y=a^2 \quad (1)$$

$$2x+y=-a^2+b^2 \quad (2)$$

$$x+2y=a^2+2c^2 \quad (3)$$

Eliminating x *and* y between (1) to (3), the resulting equation is

$$3a^2=b^2+2c^2 \quad (4)$$

which is solved through different methods and thus one obtains different sets of integer solutions to the system of equations (1) to (3).

Choice 1:

Assume $a=\alpha^2+2\beta^2$ (5)

Write 3 as $3=\left(1+i\sqrt{2}\right)\left(1-i\sqrt{2}\right)$ (6)

By substituting (5) and (6) in (4) and applying the method of factorization, define

$$\left(1+i\sqrt{2}\right)\left(\alpha+i\sqrt{2}\,\beta\right)^2=\left(b+i\sqrt{2}\,c\right)$$

Equating the real and imaginary parts , we obtain

$$\left.\begin{aligned} b &= \alpha^2-2\beta^2-4\alpha\beta \\ c &= \alpha^2-2\beta^2+2\alpha\beta \end{aligned}\right\} \quad (7)$$

Now, $(2)-(1) \Rightarrow x=b^2-2a^2$ (8)

And $(1) \Rightarrow y=3a^2-b^2$ (9)

Using the values of a, b, c given by (5) and (7) in (8) and (9), the required values of x *and* y satisfying (1) to (3) are given by

$$x = -\alpha^4 - 4\beta^4 + 4\alpha^2\beta^2 + 16\alpha\beta^3 - 8\alpha^3\beta$$

$$y = 2\alpha^4 + 8\beta^4 - 16\alpha\beta^3 + 8\alpha^3\beta$$

Denoting x & y by $x(\alpha,\beta)$ & $y(\alpha,\beta)$. A few properties observed are as follows:

Properties:

- $y(1,\beta) + 2x(1,\beta) - 6P_{\beta}^{18} - t_{4,\beta} \equiv 0 \pmod 5$
- $x(1,\alpha) + y(1,\alpha) - 4(t_{4,\alpha})^2 - 4t_{4,\alpha} - J_1$
- $x(2,\alpha) + 4Pt_{\alpha} - 60t_{4,\alpha} - 24SO_{\alpha} + 16(\alpha+1)$ is a cubical integer.
- $y(1,\alpha) - x(1,\alpha) - 12t_{4,\alpha^2} + 8t_{3,n} + 28\alpha^3 \equiv 3 \pmod{16}$
- $y(1,\beta) + 2x(1,\beta) - 8t_{4,\beta} * gn_{\beta} \equiv 0 \pmod 8$

A few numerical values are presented in Table 1 as follows:

Table 1: Numerical values

α	β	a	b	c	x	y
1	1	3	-5	1	7	2
2	2	12	-20	4	112	32
2	3	22	-38	-2	476	8
3	4	41	-71	1	1679	2

Observation 1:

From the values of x and y, one may generate second order Ramanujan numbers.

The process of obtaining the same is illustrated below.

Illustration 1: consider

$$x = 476 = 238*2 = 119*4 = 28*17 = 34*14 = 68*7$$

Now,

$$119*4=28*17 \Rightarrow 123^2+11^2=115^2+45^2=15250$$

$$28*17=68*7 \Rightarrow 45^2+61^2=11^2+75^2=5746$$

$$68*7=119*4 \Rightarrow 75^2+115^2=61^2+123^2=18850$$

$$238*2=34*14 \Rightarrow 120^2+10^2=118^2+24^2=14500$$

Thus 14500, 15250, 5746, 18850 represents second order Ramanujan numbers.

Illustration 2: consider

$$x=112=56*2=28*4=8*14$$

Now,

$$56*2=28*4 \Rightarrow 29^2+12^2=16^2+27^2=985$$

$$28*4=8*14 \Rightarrow 16^2+3^2=11^2+12^2=265$$

$$8*14=56*2 \Rightarrow 27^2+11^2=29^2+3^2=850$$

Thus 985, 265, 850 represents second order ramanujan numbers.

Choice 2:

Equation (4) can be written as

$$b^2=3a^2-2c^2 \tag{10}$$

Consider the linear transformations

$$a=X\pm 2T \ , \ c=X\pm 3T \tag{11}$$

Using (11) in (10), we have

$$X^2-b^2=6T^2 \tag{12}$$

which is satisfied by

$$\left.\begin{aligned} X&=6r^2+s^2 \\ T&=2rs \end{aligned}\right\} \tag{13}$$

$$b=6r^2-s^2 \tag{14}$$

From (11) and (13)

$$\left.\begin{aligned} a&=6r^2+s^2\pm 4rs \\ c&=6r^2+s^2\pm 6rs \end{aligned}\right\} \tag{15}$$

Employing the above values of a and b in (8) and (9), the corresponding values of x and y satisfying the system (1) to (3) are given by

$$x=-36r^4-s^4-68r^2s^2\mp 96r^3s\mp 16rs^3$$

$$y=72r^4+2s^4+96r^2s^2\pm 144r^3s\pm 24rs^3$$

A few numerical values are presented in Table 2 as follows:

Table 2: Numerical values

r	s	a	b	c	x	y
2	3	57	15	69	-6273	9522

3	2	82	50	94	-10948	17672
4	3	57	87	33	1071	2178
5	4	86	134	46	3164	4232

Observation 2:

From the values of x and y, one may generate second order Ramanujan numbers.

The process of obtaining the same is illustrated below.

Illustration 3: Consider

$$y = 9522 = 4761*2 = 3174*3 = 1587*6 = 1058*9 = 529*18$$
$$= 414*23 = 207*46 = 138*69$$

Now,

$$4761*2 = 3174*3 \Rightarrow 4763^2 + 3171^2 = 4759^2 + 3177^2 = 3274141$$
$$3174*3 = 1587*6 \Rightarrow 3177^2 + 1581^2 = 3171^2 + 1593^2 = 12592890$$
$$1587*6 = 1058*9 \Rightarrow 1593^2 + 1049^2 = 1581^2 + 1067^2 = 3638050$$
$$1058*9 = 529*18 \Rightarrow 1067^2 + 511^2 = 1049^2 + 547^2 = 1399610$$
$$529*18 = 414*23 \Rightarrow 547^2 + 391^2 = 511^2 + 437^2 = 452090$$
$$414*23 = 207*46 \Rightarrow 437^2 + 161^2 = 391^2 + 253^2 = 216890$$
$$207*46 = 138*69 \Rightarrow 253^2 + 69^2 = 161^2 + 207^2 = 68770$$

Thus 3274141, 12592890, 3638050, 1399610, 452090, 216890, 68770 represents second order Ramanujan numbers.

Illustration 4: Consider

$$x = 3164 = 1582*2 = 791*4 = 226*14 = 28*113$$

Now,

$$1582*2 = 226*14 \Rightarrow 792^2 + 106^2 = 790^2 + 120^2 = 638500$$
$$791*4 = 28*113 \Rightarrow 795^2 + 85^2 = 787^2 + 141^2 = 639250$$

Thus 638500, 639250 represents second order Ramanujan numbers.

However, there are other choices of solutions to (4) leading to different sets of solutions to the system of equations under consideration. For this, (12) is expressed as the system of double equations as given below:

Table 3 : System of double equations

System	I	II	III	IV

$X+b$	T^2	$3T^2$	$3T$	$6T$
$X-b$	6	2	$2T$	T

Consider System I:

Solving for X and b, we have

$$X=\frac{T^2+6}{2}\ ,\ b=\frac{T^2-6}{2}$$

Taking $T=2\alpha$, one obtains

$$\begin{aligned} X&=2\alpha^2+3\\ b&=2\alpha^2-3 \end{aligned} \tag{16}$$

Using the values of X and T in (11), the values of a and c are found to be

$$\left.\begin{aligned} a&=2\alpha^2\pm4\alpha+3\\ c&=2\alpha^2\pm6\alpha+3 \end{aligned}\right\} \tag{17}$$

By substituting the above equations in (8) and (9), the required values of x and y are given by

$$x=-4\alpha^4\mp32\alpha^3-68\alpha^2\mp48\alpha-9$$

$$y=8\alpha^4\pm48\alpha^3+96\alpha^2\pm72\alpha+18$$

Denoting x & y by $x(\alpha)$ & $y(\alpha)$. A few properties observed are as follows:

Properties:

- $x(\alpha)+4\left\{t_{4,\alpha^2}+14t_{4,\alpha}\right\}+24P_\alpha^{10}\equiv-9(\mathrm{mod}\,68)$
- $y(\alpha)+2x(\alpha)+4\left\{P_\alpha^3+6P_\alpha^5-2t_{4,\alpha}\right\}\equiv0(\mathrm{mod}\,16)$
- $y(\alpha)-8Pt_\alpha+\alpha(J_6+J_3)+(J_6-J_3)$ is a Perfect Square.
- Each of the following expressions represents a Cubical integer:
  - $y(\alpha)-x(\alpha)-8t_{4,\alpha}-9$
  - $4Pt_\alpha-x(\alpha)-y(\alpha)-16t_{4,\alpha}+3J_3$

A few numerical values are presented in Table 4 as follows:

Table 4: Numerical values

α	a	b	c	x	y
5	73	47	83	-8449	13778
3	33	15	39	-1953	3042
9	129	159	111	-8001	24642

8	163	125	179	-37513	64082

Observation 3:

From the values of x and y, one may generate second order Ramanujan numbers.

The process of obtaining the same is illustrated below.

Illustration 5: consider

$$y = 24642 = 12321*2 = 8214*3 = 4107*6 = 2738*9$$

Now,

$$12321*2 = 8214*3 \Rightarrow 12323^2 + 8211^2 = 12319^2 + 8217^2 = 219276850$$
$$8214*3 = 4107*6 \Rightarrow 8217^2 + 4101^2 = 8211^2 + 4113^2 = 84337290$$
$$4107*6 = 2738*9 \Rightarrow 4113^2 + 2729^2 = 4101^2 + 2747^2 = 24364210$$
$$2738*9 = 12321*2 \Rightarrow 2742^2 + 12319^2 = 2729^2 + 12323^2 = 159303770$$

Thus 219276850, 84337290, 24364210, 159303770 represents second order Ramanujan numbers.

System II:

$$X = \frac{3T^2+2}{2}, \; b = \frac{3T^2-2}{2}$$

Following the procedure as above, the values of $a, b, c, x, and\ y$ satisfying (1) to (3) are given by

$$a = 6\alpha^2 + 4\alpha + 1$$
$$b = 6\alpha^2 - 1$$
$$c = 6\alpha^2 + 6\alpha + 1$$

$$x = -36\alpha^4 \mp 96\alpha^3 - 68\alpha^2 \mp 16\alpha - 1$$
$$y = 72\alpha^4 \pm 144\alpha^3 + 96\alpha^2 \pm 24\alpha + 2$$

System III:

$$X = \frac{5T}{2}, \; b = \frac{T}{2}$$

In this case, we have

$$a = 9\alpha$$
$$b = \alpha$$
$$c = 11\alpha$$

$$x = -161\alpha^2, \; -\alpha^2$$
$$y = 242\alpha^2, \; 2\alpha^2$$

System IV:

$$X = \frac{7T}{2}, \; b = \frac{5T}{2}$$

For this choice, one obtains

$$a = 11\alpha$$
$$b = 5\alpha$$
$$c = 13\alpha$$

$$x = -217\alpha^2, \; 7\alpha^2$$
$$y = 338\alpha^2, \; 2\alpha^2$$

Choice 3:

(12) can be written as

$$b^2 + 6T^2 = X^2 * 1 \tag{18}$$

Write 1 as

$$1 = \frac{\left(1 + i2\sqrt{6}\right)\left(1 - i2\sqrt{6}\right)}{25} \tag{19}$$

Assume $X = \alpha^2 + 6\beta^2$ (20)

Substituting (19) and (20) in (18) and applying the method of factorization, define

$$\left(b + i\sqrt{6}T\right) = \frac{\left(1 + i2\sqrt{6}\right)}{5}\left(\alpha + i\sqrt{6}\,\beta\right)^2$$

Equating the real and imaginary parts in the above equation, we obtain

$$\left.\begin{aligned} b &= \frac{1}{5}\left(\alpha^2 - 6\beta^2 - 24\alpha\beta\right) \\ T &= \frac{1}{5}\left(2\alpha^2 - 12\beta^2 + 2\alpha\beta\right) \end{aligned}\right\} \tag{21}$$

Since our aim is to find integer solutions, substituting $\alpha = 5A$, $\beta = 5B$ in (20) and (21) ,we get

$$X = 25A^2 + 150B^2$$

$$\left.\begin{aligned} b &= 5A^2 - 30B^2 - 120AB \\ T &= 10A^2 - 60B^2 + 10AB \end{aligned}\right\},$$

After performing some algebra, one obtains

$$a = 45A^2 + 30B^2 + 20AB$$

$$b = 5A^2 - 30B^2 - 120AB$$

$$c = 55A^2 - 30B^2 + 30AB$$

$$x = -4025A^4 - 900B^4 + 7900A^2B^2 + 4800AB^3 - 4800A^3B$$

$$y = 6050A^4 - 1800B^4 - 4800A^2B^2 - 3600AB^3 + 6600A^3B$$

Denoting x & y by $x(A, A)$ & $y(A, A)$. A few properties observed are as follows:

Properties:

- $y(1, A) + x(1, A) - 3100t_{4,A} - 1800\,OH_A - 1200A$ is the sum of two squares.
- $x(A, A) + y(A, A)$ is a perfect square.

4. CONCLUSION:

To conclude, one may search for other sets of integer solutions for the system of triple equations under consideration.

REFERENCES:

1) *Dr.S.Vidhyalakshmi, T.Mahalakshmi, Dr.J.Shanthi, Dr.M.A.Gopalan, "On Two Interesting systems of Diophantine Equations", Journal of Interdisciplinary Cycle Research, 11(11), Nov-(2019), Pp.692-695.*
2) *Dr.S.Vidhyalakshmi, T.Mahalakshmi, H.Ayesha Begum, A.Prathiba, Dr.M.A.Gopalan, "On Two Interesting Systems of Triple Diophantine Equations with Five Unknowns", IJRAR, 6(2), June (2019), Pp.637-645.*
3) *Dr.A.Vijayasankar, Sharadha kumar, Dr.M.A.Gopalan, "On the system of Triple Equations with five variables* $x+y=a^2, 2x+y=b^2, x+2y=a^2-c^2$*", Journal of scientific computing, a(1),(2020) Pp:26-28.*
4) *Dr.N.Thiruniraiselvi, J.Mathumitha, Dr.M.A.Gopalan, "On the Integral solutions of Triple Equations* $x+y=a^2$, $2x+y=a^2+b^2$, $x+2y=a^2+5c^2$*", Compliance Engineering Journal 11(3), (2020), Pp:71-77.*
5) *Dr.N.Thiruniraiselvi, J.Mathumitha, Dr.M.A.Gopalan, " On Simultaneous Equations* $x+y=a^2$, $2x+y=a^2+3b^2$, $x+2y=a^2+c^2$*", Science, Technology and Development IX(III), (2020), Pp:132-136.*