A Distributed delay Model with One Ammensal and Two Mutualistic Species

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Abstract: *In this paper we studied the dynamics of one ammensal and two mutualistic species. A distributed time lag is induced in the interaction of ammensal and the second mutual species. Local and global stability analysis is discussed at co-existing state. Numerical simulation with different delay kernel strengths are illustrated and proved that delay kernels have no impact in the population dynamics of two mutual species.*

Keywords:Ammensal, Mutualism, Co-existing state, Stability, delay kernels.

1. INTRODUCTION

The research in ecology using mathematical tools like differential equations plays a vital role in stability analysis of eco systems. Ecological interactions may be classified like prey-predation, competition ammensalism, commensalism, and mutualism etc. Sometimes the models arise from the combination of two or three classifications.Research in this discipline was initiated by Lokta[1] and Voltera [2]. Stability analysis of ecological systems were widely discussed by May [3] , Freedman [5] and Kapur [6,7].

One of such situation is discussed in this paper and studied the dynamics of the model with one ammensal and two mutual species. In ammesalism one has adverse effect with other living being. Ammesal will not get any benefit or loss while others will get negative effect. We also induced a distributed time lag in the interaction of ammensal and the first mutualistic species. Delays are common in ecological systems. Distributed time lags are more appropriate to use in ecological systems. Ecological interaction with distributed lags are explained by Cushing [4], Sreeharirao[15] and yang[16]. Lakshmi Narayan et.al [9]studied three species model with prey, predator and ammensalmodels. Kondalarao [8] discussed a three specie dynamical system of ammensal relationship of humans on plants and birds with time delay. Distributed type time delay models with prey, predator and competitor models were discussed by Paparao [11,13,14]. Distributed type of delay in three species ammensalism model was dealt by Paparao [12]. In continuation with we proposed an ecological model with one ammensal and two mutual helping species. A distributed type delay is induced in the interaction of ammensal and second mutual species. The dynamics of the model studied with different delay kernel strengths and observed that delay arguments has no impact in the population strengths of two mutual species when no delay arguments are included. So the delay arguments are not significant in the population dynamics of the two mutual species population.

2. Mathematical Model

The mathematical model equations for the proposed model(logistic growth model) with distributed time lag in the interaction of ammensal and the second mutual species is given by the following equations.

$$\frac{dx}{dt} = a_1 x\left[1-\frac{x}{c_1}\right]$$

$$\frac{dy}{dt} = a_2 y\left[1-\frac{y}{c_2}\right] - \alpha_{21} yx + \alpha_{23} yz \qquad (2.1)$$

$$\frac{dz}{dt} = a_3 z\left[1-\frac{z}{c_3}\right] - \alpha_{31} z\int_{-\infty}^{t} w(t-u)\, x(u) du + \alpha_{32} zy$$

Where
x- is the Ammensal population,
y & z are the mutualistic species populations,
a_1, a_2 *and* a_3 are the natural growth rate of the ammensal and mutualistc species,
α_{21} & α_{31} are the rate of decay of mutualistic species due to attacks of ammensal species.
α_{23} & α_{32} are the rate of growth of mutulistic species due to helping one each other.
c_1, c_2 *and* c_3 are the carrying capacities of the ammensal and mutualistic species.

Further the variables *x, y, and z* are non-negative and the model parameters $a_1, a_2, a_3, \alpha_{21}, \alpha_{23}, \alpha_{31}$ and α_{32} are assumed to be non negative constants.

Let us take $\frac{a_1}{c_1} = k_1, \frac{a_2}{c_2} = k_2, \frac{a_3}{c_3} = k_3$.

Put t-u = s , we get the following system of equations

$$\frac{dx}{dt} = a_1 x\left[1-\frac{x}{c_1}\right]$$

$$\frac{dy}{dt} = a_2 y\left[1-\frac{y}{c_2}\right] - \alpha_{21} yx + \alpha_{23} yz \qquad (2.2)$$

$$\frac{dz}{dt} = a_3 z\left[1-\frac{z}{c_3}\right] - \alpha_{31} z\int_{0}^{\infty} w(s)\, x(t-s) ds + \alpha_{32} zy$$

Choose the kernel *w* such that

$$\int_0^\infty w(s)ds = 1, \int_0^\infty sw(s)ds < \infty, \qquad (2.3)$$

3. Equilibrium Point:

The system under investigation, eight equilibrium points are identified. Out of these we studied only on co-existing state which is given by

$$\bar{x} = c_1$$

$$\bar{y} = \frac{c_1(\alpha_{23}\alpha_{31}+k_3\alpha_{21})-(a_3\alpha_{23}+a_2k_3)}{(\alpha_{23}\alpha_{32}-k_2k_3)} \qquad (3.1)$$

$$\bar{z} = \frac{c_1(k_2\alpha_{31}+\alpha_{21}\alpha_{32})-(a_3k_2+a_2\alpha_{32})}{(\alpha_{23}\alpha_{32}-k_2k_3)}$$

This state would exist only when

$$\left(c_1(\alpha_{23}\alpha_{31}+k_3\alpha_{21}) > (a_3\alpha_{23}+a_2k_3)\right), c_1(k_2\alpha_{31}+\alpha_{21}\alpha_{32}) > (a_3k_2 + a_2\alpha_{32})) \text{ and } (\alpha_{23}\alpha_{32}-k_2k_3) > 0 \qquad (3.2)$$

4. Stability of the Co-existing State:

Theorem: The co-existing state E$(\bar{x},\bar{y},\bar{z})$ is locally asymptotically stableIf $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$

Proof: Let the variational matrix is givenby

$$J = \begin{bmatrix} -k_1\bar{x} & 0 & 0 \\ -\alpha_{21}\bar{y} & -k_2\bar{y} & \alpha_{23}\bar{y} \\ -\alpha_{31}\bar{z}w(s) & \alpha_{32}\bar{z} & -k_3\bar{z} \end{bmatrix} \qquad (4.1)$$

The characteristic equation of the system is$|\lambda I - J| = 0$

$$\Rightarrow \lambda^3 + \lambda^2 b_1 + \lambda\, b_2 + b_3 = 0 \qquad (4.2)$$

Where$b_1 = k_1\bar{x} + k_2\bar{y} + k_3\bar{z} > 0$,

$$b_2 = k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z} + k_2\, k_3\bar{y}\bar{z} - \alpha_{23}\,\alpha_{32}\bar{y}\bar{z}$$

$$\Rightarrow b_2 = k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z} + (k_2\, k_3 - \alpha_{23}\,\alpha_{32})\bar{y}\bar{z} \quad \text{and}$$

$$b_3 = k_1\bar{x}(k_2\, k_3\bar{y}\bar{z} - \alpha_{23}\,\alpha_{32}\bar{y}\bar{z}) \Rightarrow b_3 = k_1\bar{x}\bar{y}\bar{z}(k_2\, k_3 - \alpha_{23}\,\alpha_{32})$$

⇨ $b_1b_2 - b_3 = (k_1\bar{x} + k_2\bar{y} + k_3\bar{z})(k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z} + (k_2\, k_3 - \alpha_{23}\,\alpha_{32})\bar{y}\bar{z}) - k_1\bar{x}\bar{y}\bar{z}(k_2\, k_3 - \alpha_{23}\,\alpha_{32})$

⇨ $b_1b_2 - b_3 = (k_1\bar{x} + k_2\bar{y} + k_3\bar{z})(k_1\, k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z}) + (k_2\bar{y} + k_3\bar{z})(k_2\, k_3 - \alpha_{23}\,\alpha_{32})\bar{y}\bar{z}$

⇨ $b_1b_2 - b_3 > 0$ if $k_2\, k_3 > \alpha_{23}\alpha_{32}$ and

⇨ $b_3(b_1b_2 - b_3) = k_1\bar{x}\bar{y}\bar{z}(k_2 k_3 - \alpha_{23}\alpha_{32})((k_1\bar{x} + k_2\bar{y} + k_3\bar{z})(k_1 k_2\bar{x}\bar{y} + k_1k_3\bar{x}\bar{z}) + (k_2\bar{y} + k_3\bar{z})(k_2 k_3 - \alpha_{23}\alpha_{32})\bar{y}\bar{z})$

Which is positive if $k_2 k_3 > \alpha_{23}\alpha_{32}$

We have $b_1 > 0$, $(b_1b_2 - b_3) > 0$ and $b_3(b_1b_2 - b_3) > 0$ if $k_2 k_3 > \alpha_{23}\alpha_{32}$

Therefore by Routh – Hurwitz criteria, the system is Asymptotically stable if $k_2 k_3 > \alpha_{23}\alpha_{32}$

Hence the co-existing state $(\bar{x}, \bar{y}, \bar{z})$ is locally asymptotically stable if $k_2 k_3 > \alpha_{23}\alpha_{32}$

5. Global Stability:

Statement: The co-existing state is globally asymptotically stable.

Proof: Let us choose theLyapunov's function

$$V(\bar{x},\bar{y},\bar{z}) = \left\{x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right\} + \left\{y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right\} + \left\{z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right\} \quad (5.1)$$

Here $\bar{x} \neq 0, \bar{y} \neq 0, \bar{z} \neq 0$

Differentiate (5.1) with respect to't', we get

$$\frac{dV}{dt} = \left[\frac{x-\bar{x}}{x}\right]\frac{dx}{dt} + \left[\frac{y-\bar{y}}{y}\right]\frac{dy}{dt} + \left[\frac{z-\bar{z}}{z}\right]\frac{dz}{dt} \quad (5.2)$$

$$\Rightarrow \frac{dV}{dt} = \left\{\begin{array}{c}\left[\frac{x-\bar{x}}{x}\right](a_1x - k_1x^2) + \left[\frac{y-\bar{y}}{y}\right](a_2y - k_2y^2 - \alpha_{21}yx + \alpha_{23}yz) \\ + \left[\frac{z-\bar{z}}{z}\right](a_3z - k_3z^2 - \alpha_{31}zxw(\lambda) + \alpha_{32}zy)\end{array}\right\} \quad (5.3)$$

$$\Rightarrow \frac{dV}{dt} = \left\{\begin{array}{c}\left[\frac{x-\bar{x}}{x}\right](a_1x - k_1x^2) + \left[\frac{y-\bar{y}}{y}\right](a_2y - k_2y^2 - \alpha_{21}yx + \alpha_{23}yz) \\ + \left[\frac{z-\bar{z}}{z}\right](a_3z - k_3z^2 - \alpha_{31}z\int_0^\infty w(s)\,x(t-s)ds + \alpha_{32}zy)\end{array}\right\}$$

$$\Rightarrow \frac{dV}{dt} = \left\{\begin{array}{l}[x-\bar{x}](a_1 - k_1x) + [y-\bar{y}](a_2 - k_2y - \alpha_{21}x + \alpha_{23}z) \\ +[z-\bar{z}](a_3 - k_3z - \alpha_{31}\int_0^\infty w(s)\,x(t-s)ds + \alpha_{32}y)\end{array}\right\} \quad (5.4)$$

Choosing $a_1 = k\bar{x}, a_2 = k_2\bar{y} + \alpha_{21}\bar{x} + \alpha_{23}\bar{z}, a_3 = k_2\bar{z} + \alpha_{31}\int_0^\infty w(s)x(t-s)ds + + \alpha_{32}\bar{y}.$

$= -k_1(x-\bar{x})^2 + (y-\bar{y})[-\alpha_{21}(x-\bar{x}) - k_2(y-\bar{y})] + (z-\bar{z})[-\alpha_{31}(x-\bar{x}) - k_3(z-\bar{z})]$

$$= -k_1(x-\bar{x})^2 - k_2(y-\bar{y})^2 - k_3(z-\bar{z})^2 - \alpha_{21}(x-\bar{x})(y-\bar{y}) - \alpha_{31}(x-\bar{x})(z-\bar{z}) \quad (5.5)$$

Using the basic inequality $ab \leq \frac{a^2 + b^2}{2}$

$$= -k_1(x-\bar{x})^2 - k_2(y-\bar{y})^2 - k_3(z-\bar{z})^2 - \frac{\alpha_{21}}{2}[(x-\bar{x})^2 + (y-\bar{y})^2] - \frac{\alpha_{31}}{2}[(x-\bar{x})^2 + (z-\bar{z})^2] \quad (5.6)$$

$$= -(x-\bar{x})^2\left[k_1 + \frac{\alpha_{21}}{2} + \frac{\alpha_{31}}{2}\right] - (y-\bar{y})^2\left[k_2 + \frac{\alpha_{21}}{2}\right] - (z-\bar{z})^2\left[k_3 + \frac{\alpha_{31}}{2}\right] \quad (5.7)$$

$$\Rightarrow \frac{dV}{dt} < 0 \quad (5.8)$$

Hence the Normal steady state is globally asymptotically stable.

6. Numerical Examples:

S.No	Figures	Description
1	The figures(A)	Shows the variation of x, y and z with respect to Time (t)
2	The figures(B)	The phase portrait of x, y and z

Example 6.1: $a_1 = 0.2$; $a_2 = 0.5$; $a_3 = 0.2$; α_{21}=0.05; α_{23}=0.05; α_{31}=0.05; α_{32}=0.05; c_1 =50; c_2=50; c_3 =50, x=10, y=5, z=2.

The system is asymptotically stable to E(60, 97,86) when no delay arguments are induced .

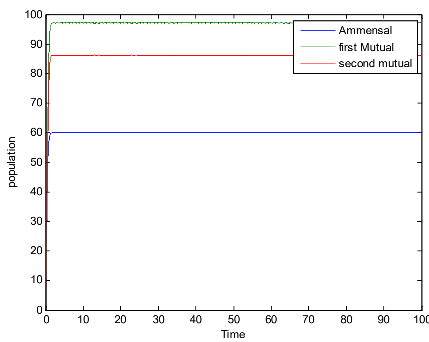


Figure:6. 1(A)

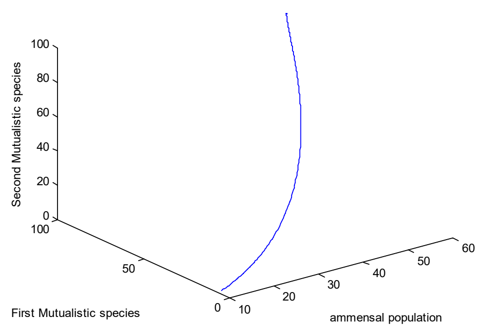
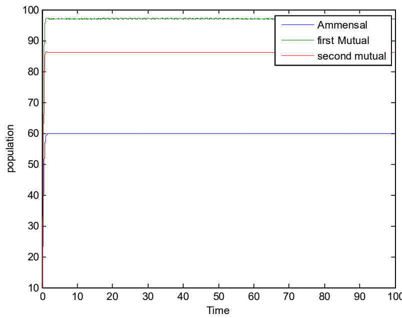


Figure: 6.1(B)

With the kernels as follows $w(s) = ae^{-as}$ $for\ a > 0$, and the Laplace transform of $w(s)$ is defined as $w(\boldsymbol{\lambda}) = \int_0^\infty \boldsymbol{e^{-\lambda t} a\, e^{-at} dt} = \frac{\boldsymbol{a}}{\boldsymbol{a+\lambda}}$

The results are simulated for the above system of equations (2.2) Using MAT LAB simulation.With the parameters shown in Example 6. 1 with different kernel values are plotted below.

1. a=0.1; λ=2; (60, 97, 86)



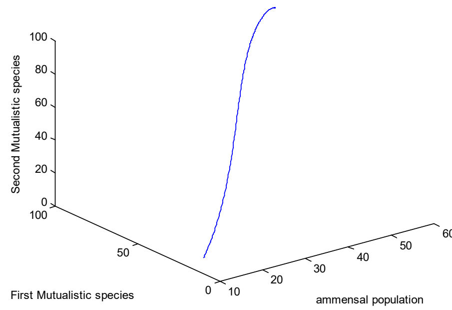
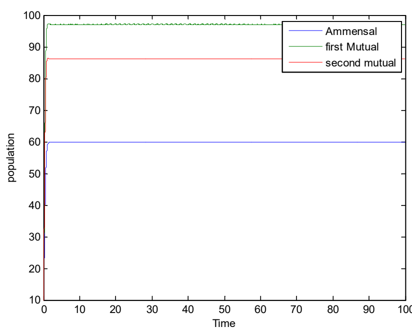


Figure:6. 1.1(A) *Figure*: 6.1.1(B)

The system is asymptotically stable to E(60, 97,86) . No significant growth is observed in two mutual species.

2. a=0.1; λ=0.2; (60,97,86)



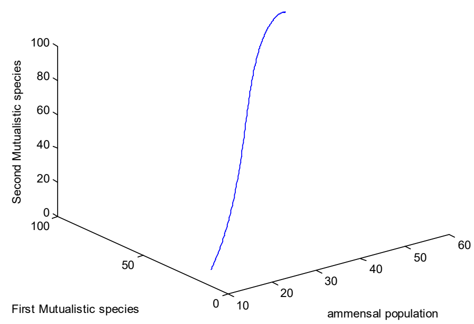
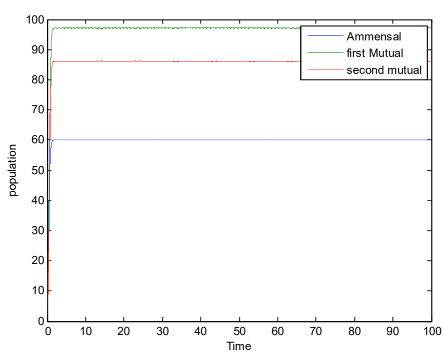


Figure:6. 1.2(A) *Figure*: 6.1.2(B)

The system is asymptotically stable to E(60, 97,86) . No significant growth is observed in two mutual species.

3. a=1; λ=0.2; (60, 9786)



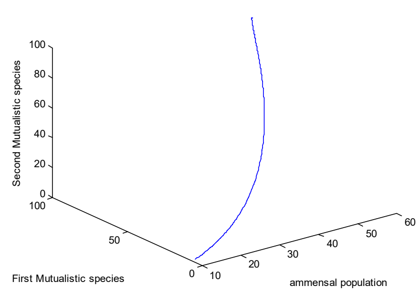
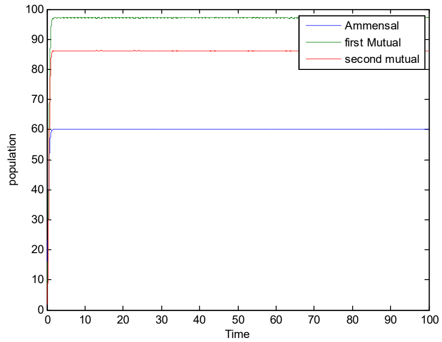


Figure:6. 1.3(A) *Figure*: 6.1.3(B)

The system is asymptotically stable to E(60, 97,86) . No significant growth is observed in two mutual species.

4. a=1; λ=2; (60, 97, 86)



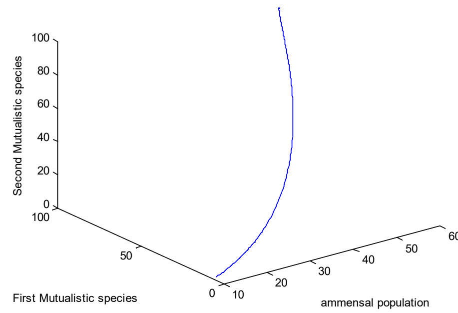
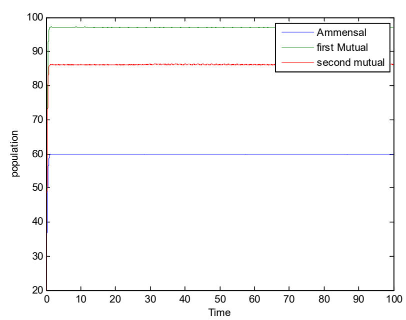


Figure:6. 1.4(A) *Figure*: 6.1.4(B)

The system is asymptotically stable to E(60, 97,86) . No significant growth is observed in two mutual species.

Example 6. 2:a_1 =1; a_2 =2; a_3 =3;α_{21}=0.05; α_{23}=0.03;α_{31}=0.05;α_{32}=0.03;
c_1 =25;c_2 =25; c_3 =25, x=20, y=20, z=20.



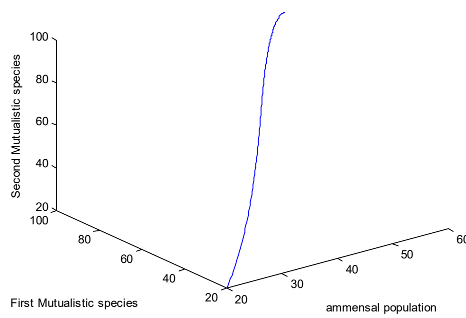
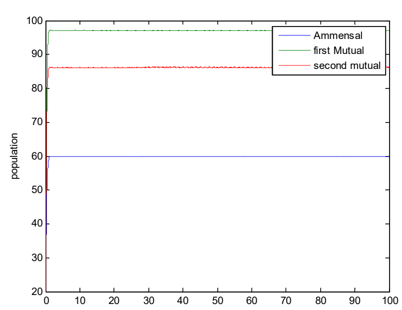


Figure:6.2(A) *Figure*: 6.2(B)

The system is stableto E (60, 97, 86) when no delay arguments are induced.

When delay is induced with different values of λ and a isgiven below.

1. a=0.1; λ=2; (60, 97,86)



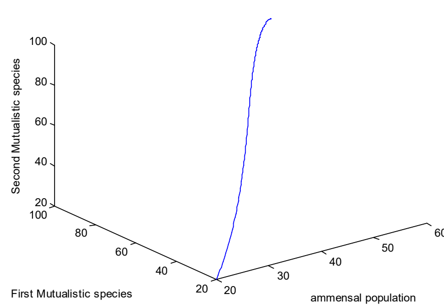


Figure:6.2.1(A) *Figure*: 6.2.1(B)

System is stable toE(60, 97.06, 86.21). No Significant growth is observed in two mutual species when compare with no delay argument.

2. a=0.1; λ=0.2; (60,97.06,86.21)

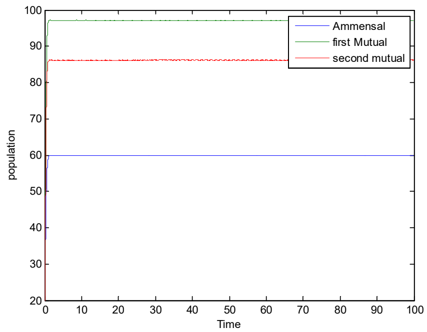


Figure:6.2.2(A)

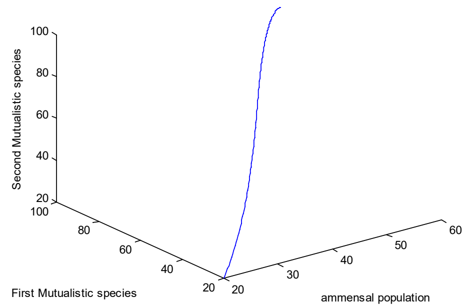


Figure: 6.2.2(B)

System is stable to E(60, 97 , 86). No significant growth is observed in first and second mutual species.

3. a=1; λ=0.2; (60, 97, 86)

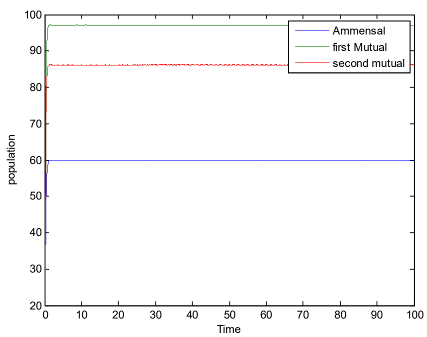


Figure:6.2.3(A)

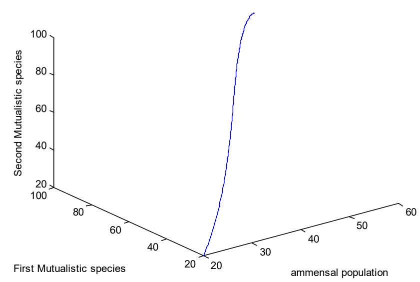


Figure: 6.2.3(B)

System is stable to E(60, 97, 86). No significant growth is observed in first and second mutual species when compare with no delay argument in the system.

4. a=1; λ=2; (60, 97, 86)

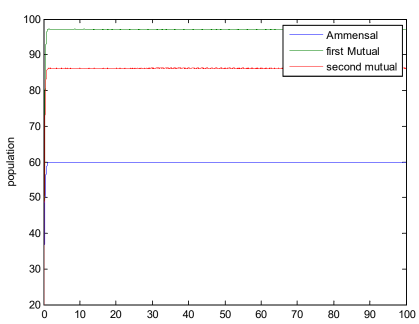


Figure:6.2.4(A)

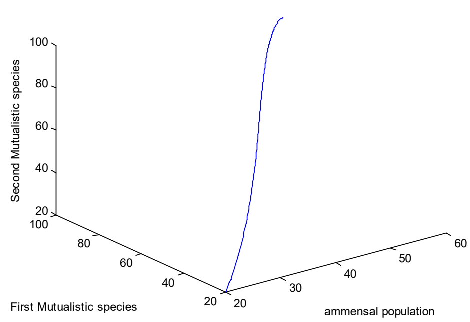


Figure: 6.2.4(B)

System is stable to E(60, 97, 86). No Significant growth is observed in first and second mutual species when compare with no delay argument in the sytem.

Example :6.3 a_1 =5; a_2 =6; a_3 =7; α_{21}=0.5; α_{23}=0.5;α_{31}=0.5;α_{32}=0.5;c_1=60;c_2=60; c_3 =60; x=50, y=50, z=50.

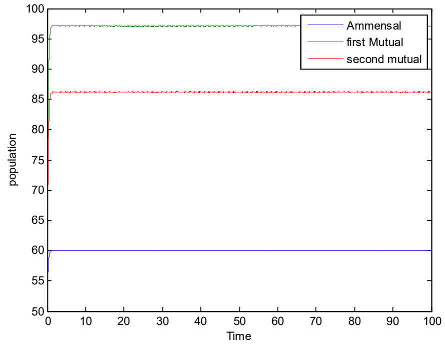


Figure:6.3(A)

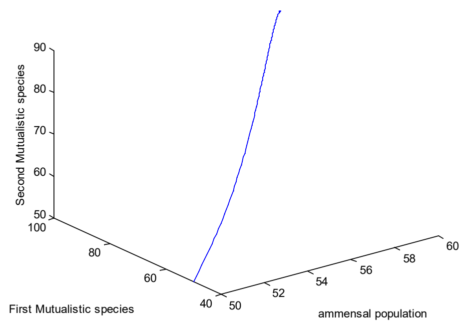


Figure: 6.3(B)

System is stable to E(60, 97, 86) when no delay arguments are induced in the system

1 a=0.1; λ=2; (60, 97, 86)

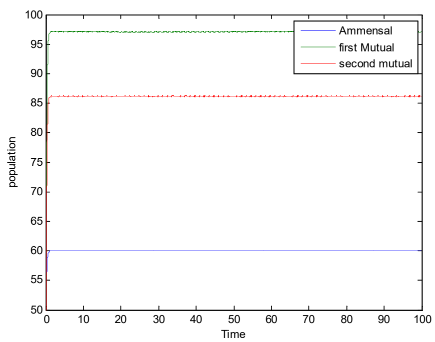


Figure:6.3.1(A)

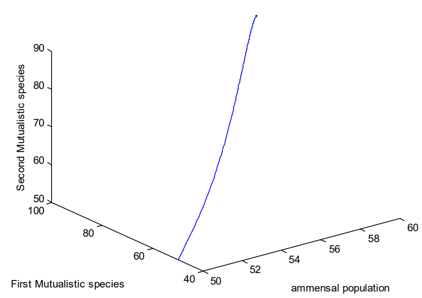


Figure: 6.3.1(B)

System is stable to E(60, 97, 86). No significant growth is observed in first and second mutual species when compare with no delay argument.

2. a=0.1; λ=0.2; (60, 97, 86)

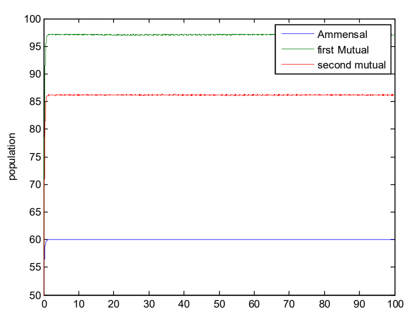


Figure:6.3.2(A)

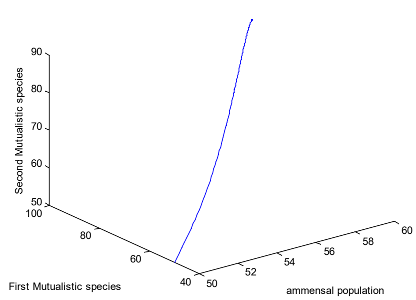


Figure: 6.3.2(B)

System is stable to E(60, 97 , 86). No significant growth is observed in first and second mutual species when compare with no delay argument.

3. a=1; λ=0.2; (60, 97, 86)

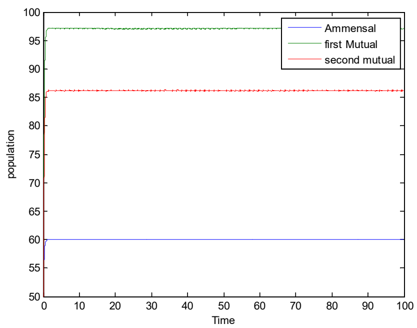


Figure:6.3.3(A)

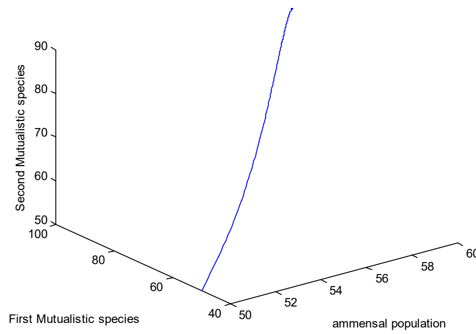


Figure: 6.3.3(B)

System is stable to E(60, 97, 86). No significant growth is observed in first and second mutual species when compare with no delay argument.

4. a=1; λ=2; (60, 97, 86)

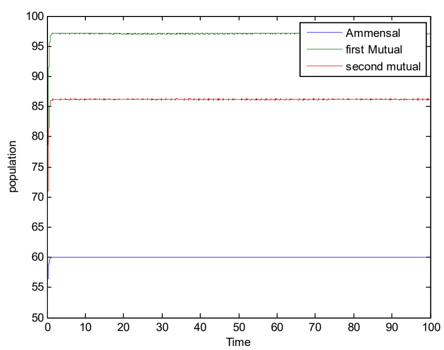


Figure:6.3.4(A)

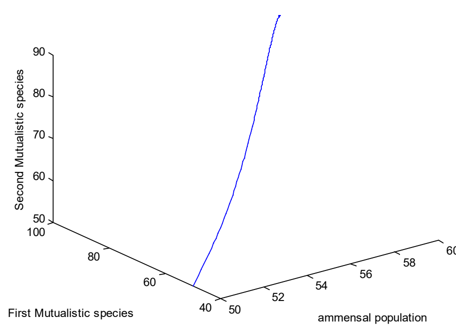


Figure: 6.3.4(B)

System is stable to E(60, 97, 86). No Significant growth is observed in first and second mutual species when compare with no delay argument.

7. Conclusion:

We consider a three species ecological model in which the ammensal and two mutualistic species.The distributed time lag is induced in the interaction of ammensal and the second mutual species. The co-existing state is identified and studied the local stability analysis at this point and shown that the system is asymptotically stable if $k_2\, k_3 > \alpha_{23}\, \alpha_{32}$ .The global stability is studied by lyapunov's function. The dynamics of the system is studied using numerical simulation in support of stability analysis. We consider three numerical examples with delay and without delay arguments. The impact of delay with different kernel strength is studied and observed that the systemis stable and delay arguments have no significant role in system dynamics. The population strengths when compared with no delay arguments are unchanged. So the delay arguments have no impact in the population of two mutually helping species.

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