On the Ternary Quadratic Equation

$(a+1)x^2 - ay^2 = z^2,\ a > 0$

Dr. Shreemathi Adiga
Asst professor of Mathematics
Govt First Grade College, Koteshwara Kundapura, Udupi District, Karnataka, INDIA.

Abstract:

The homogeneous ternary quadratic Diophantine equation representing the cone $(a+1)x^2 - ay^2 = z^2,\ a > 0$ is analyzed for its patterns of non-zero distinct integral solutions.

Keywords- Ternary quadratic, Homogeneous quadratic,Integral solutions.

Introduction:

The Diophantine equation offers an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-15] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $(a+1)x^2 - ay^2 = z^2,\ a > 0$ representing homogeneous quadratic with three unknowns for determining its infinitely many non-zero integral points.

Method of analysis:

The ternary quadratic equation to be solved is

$$(a+1)x^2 - ay^2 = z^2,\ a > 0 \tag{1}$$

Different choices of solutions to (1) are illustrated below:

Choice 1:

Using the linear transformations

$$x = X + aT,\ y = X + (a+1)T \tag{2}$$

in (1), it reduces to

$$X^2 - a(a+1)T^2 + z^2 \tag{3}$$

which is is a Diophantine equation of the type $Z^2 = Dx^2 + y^2$ whose solutions are given by x = 2mn,

$y = Dm^2 - n^2$, $z = Dm^2 + n^2$

So (3) is satisfied by

$$X = a(a+1)m^2 + n^2,\ T = 2mn \tag{4}$$

$$z = a(a+1)m^2 - n^2 \tag{5}$$

Using (4) in (2), note that

$$\left.\begin{aligned} x &= (a^2+a)m^2+n^2+2mna \\ y &= a(a+1)m^2+n^2+(a+1)2mn \end{aligned}\right\} \quad (6)$$

Thus, (5) and (6) represent the integer solutions to (1).

Also, there are other solutions to (3). For obtaining the same, express (3) as the system of double equations as shown in Table 1 below:

Table 1: System of double equations

System	I	II	III	IV	V
$X+z$	$(a+1)T$	T^2	$(a+1)T^2$	aT^2	$a(a+1)T$
$X-z$	aT	$a(a+1)$	a	$(a+1)$	T

Solving each of the system of double equation in Table 1, the values of X, T and z are obtained. Substituting the values of X and T in (2), the corresponding values of x and y are determined. Thus, the above values of x, y and z satisfying (1).

For simplicity and brevity, the integer solutions to (1) obtained through solving each of the system of double equations in table 1 are exhibited below:

Solutions for system I:

$$X + Z = (a+1)\,T$$

$$X - Z = a\,T$$

$$X = \frac{(2a+1)T}{2}$$

$$Z = \frac{T}{2}$$

For Z to be an integer, we must have $T = 2p$

$$\left.\begin{aligned} X &= (2a+1)\,p, \\ z &= {}^{2p}/_{2} = p \end{aligned}\right\} \quad p \in I$$

$$x = X + a\,T \quad = (2a+1)\,p + a\,(2p) = (4a+1)\,p$$

$$y = X + (a+1)T = (2a+1)\,p + (a+1)\,2p = (4a+3)\,p$$

Thus a set of solutions for system (1 is

$$x = (4a+1)p$$
$$y = (4a+3)p$$
$$z = p$$

Similarly we obtain the solution for other systems

Solutions of system II:

$$x = 2s^2 + 2as + t_{3,a}$$
$$y = 2s^2 + 2(a+1)s - t_{3,a}$$
$$z = 2s^2 - t_{3,a}$$
$$\text{where } t_{3,a} = \frac{a(a+1)}{2}$$

Solutions of system III:

$$x = p(4q^2+1) + 2q^2 + 4pq$$
$$y = p(4q^2+1) + 2q^2 + 2q(2p+1)$$
$$z = 2(2p+1)q^2 - p$$

Solutions of system IV:

$$x = p(4q^2+1) + 2q^2 + 1 + 2q(2p+1)$$
$$y = p(4q^2+1) + 2q^2 + 1 + 4q(p+1)$$
$$z = 2(4q^2-1) + 2q^2 - 1$$

Solutions of system V:

$$x = (a^2+a+1)p + 2ap$$
$$y = (a^2+a+1)p + 2(a+1)p$$
$$z = (a^2+a-1)p$$

Choice 2:

(1) Can also be written as

$$z^2 + ay^2 = (a+1)x^2 \quad (7)$$

Assume that $x = \alpha^2 + a\beta^2$, $\alpha, \beta \neq 0$

$= (\alpha + i\sqrt{a}\,\beta)(\alpha - i\sqrt{a}\,\beta)$ (8)

Writing $z^2 + ay^2 = (z + i\sqrt{a}\,y)(z - i\sqrt{a}\,y)$

$$a + 1 = (1 + i\sqrt{a})\,(1 - i\sqrt{a}) \tag{9}$$

Substituting (8), (9) in (7) and applying the method of factorization,

$$\left(z + i\sqrt{a}\,y\right) = \left(\alpha + i\sqrt{a}\,\beta\right)^2\left(1 + i\sqrt{a}\right)$$

On equating the real and imaginary parts, we have

$$\left.\begin{aligned} z &= \alpha^2 - a\beta^2 - 2a\alpha\beta \\ y &= \alpha^2 - a\beta^2 + 2\alpha\beta \end{aligned}\right\} \tag{10}$$

Thus, (8) and (10) represent the integer solutions to (1).

Conclusion

In this paper, an attempt has been made to obtain many non-zero distinct integer solutions to **theTernary Quadratic Equation $(a+1)x^2 - ay^2 = z^2$, $a > 0$**.

In addition to the linear transformations (2), one may also use other linear transformations $x = X - aT$, $y = X - (a+1)T$ giving different sets of integer solutions to (1). Since Diohantine equations are rich in variety, the researchers may search for integer solutions to the other types of Ternary quadratic equations with variables greater than or equal to three.

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