ON THE BINARY QUADRATIC EQUATION $9x^2 - 8y^2 = 49$

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Abstract: This paper deals with the problem of obtaining non-zero distinct integer solutions to the non homogeneous binary quadratic equation represented by the Pell-like equation $9x^2 - 8y^2 = 49$. Different sets of integer solutions are presented. Employing the solutions of the above equation, integer solutions for other choices of hyperbolas, parabolas and straight lines are obtained. A special Pythagorean triangle is also determined, The construction of second order Ramanujan Numbers is illustrated.

Keywords: Non homogeneous , Binary quadratic, Pell-like equation, hyperbola, parabola, integral solutions, Special numbers.

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1. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [15-20]. In [1-14] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $9x^2 - 8y^2 = 49$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited. A special Pythagorean triangle is also determined. The construction of second order Ramanujan Numbers is illustrated.

2. METHOD OF ANALYSIS

Consider the non homogeneous binary quadratic equation

$$9x^2 - 8y^2 = 49 \tag{1}$$

Introducing the linear transformations

$$x = X + 8T, y = X + 9T \tag{2}$$

in (1), it leads to $X^2 = 72T^2 + 49$ (3)

with the least positive integer solutions $X_0 = 11, T_0 = 1$

To obtain the other solutions of equation (3) ,Consider the Pellian equation

$$X^2 = 72T^2 + 1$$

whose general solution, $\tilde{X}_n = \frac{1}{2} f_n, \tilde{T}_n = \frac{1}{2\sqrt{72}} g_n$

in which $f_n = [(17 + 2\sqrt{72})^{n+1} + (17 - 2\sqrt{72})^{n+1}]$

$g_n = [(17 + 2\sqrt{72})^{n+1} - (17 - 2\sqrt{72})^{n+1}]$, where n= -1, 0, 1, 2……

Applying Brahmagupta lemma between the solutions of (X_0, T_0) and $(\tilde{X}_n, \tilde{T}_n)$ the general solutions of equation (3) are found to be

$$X_{n+1} = \frac{11}{2} f_n + \frac{\sqrt{72}}{2} g_n$$

$$T_{n+1} = \frac{1}{2} f_n + \frac{11}{2\sqrt{72}} g_n$$

In view of (2), the corresponding nonzero distinct integral solutions of (1) are

$$x_{n+1} = \frac{19f_n}{2} + \frac{80g_n}{\sqrt{72}}$$

$$y_{n+1} = 10f_n + \frac{171g_n}{2\sqrt{72}}$$

The recurrence relations satisfied by the values of x and y are respectively

$$x_{n+3} - 34x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 34y_{n+2} + y_{n+1} = 0$$

A few numerical examples are presented in the table 1 below:

Table 1: Numerical examples

n	$\mathbf{x_{n+1}}$	$\mathbf{y_{n+1}}$
-1	19	20
0	643	682
1	21843	23168
2	742019	787030
3	25206803	26735852

4	856289283	908231938

A few interesting properties are given below:

1. The values of x are odd while the values of y are even.
2. Each of the following is a Nasty number

- $\frac{6}{49}\left[342x_{2n+2}-320y_{2n+2}+98\right]$
- $\frac{6}{833}\left[11574x_{2n+2}-320y_{2n+3}+1666\right]$
- $\frac{6}{784}\left[10912\ x_{2n+2}-320\ x_{2n+3}+1568\ \right]$
- $\frac{6}{26656}[370688x_{2n+2}-320x_{2n+4}+53312]$
- $\frac{6}{28273}\left[393174x_{2n+2}-320y_{2n+4}+56546\right]$
- $\frac{6}{784}\left[370688x_{2n+3}-10912x_{2n+4}+1568\right]$
- $\frac{6}{833}\left[342x_{2n+3}-10912y_{2n+2}+1666\right]$
- $\frac{6}{49}\left[11574x_{2n+3}-10912y_{2n+3}+98\right]$
- $\frac{6}{833}\left[393174x_{2n+3}-10912y_{2n+3}+1666\right]$
- $\frac{6}{28273}\left[343x_{2n+4}-370688y_{2n+2}+56546\right]$
- $\frac{6}{833}\left[11574x_{2n+4}-370688y_{2n+3}+1666\right]$

3. Each of the following is a Square number

- $\frac{1}{49}\left[393174x_{2n+4}-370688y_{2n+4}+98\right]$

- $\frac{1}{882}\left[342y_{2n+3}-11574y_{2n+2}+1764\right]$
- $\frac{1}{29988}\left[342y_{2n+4}-393174y_{2n+2}+59976\right]$
- $\frac{1}{882}\left[11574y_{2n+4}-393174y_{2n+3}+1764\right]$

4. Each of the following is a cubical integer

- $\frac{1}{784}\left[10912x_{3n+3}-320x_{3n+4}+32736x_{n+1}-960x_{n+2}\right]$
- $\frac{1}{26656}\left[370688x_{3n+3}-320x_{3n+5}+1112064x_{n+1}-960x_{n+3}\right]$
- $\frac{1}{49}\left[324x_{3n+3}-320y_{3n+3}+34722x_{n+2}-32736y_{n+2}\right]$
- $\frac{1}{833}\left[11574x_{3n+3}-320y_{3n+4}+1026x_{n+2}-32736y_{n+1}\right]$
- $\frac{1}{28273}\left[393174x_{3n+3}-320y_{3n+5}+1029x_{n+3}-1112064y_{n+1}\right]$
- $\frac{1}{784}\left[370688x_{3n+4}-10912x_{3n+5}+1112064x_{n+2}-32736x_{n+3}\right]$
- $\frac{1}{833}\left[342x_{3n+4}-10912y_{3n+3}+34722x_{n+1}-960y_{n+2}\right]$
- $\frac{1}{49}\left[11574x_{3n+4}-10912y_{3n+4}+1026x_{n+1}-960y_{n+1}\right]$
- $\frac{1}{833}\left[393174x_{3n+4}-10912y_{3n+4}+1026x_{n+2}-32736y_{n+1}\right]$
- $\frac{1}{28273}\left[343x_{3n+5}-370688y_{3n+3}+1179522x_{n+1}-960y_{n+3}\right]$
- $\frac{1}{833}\left[11574x_{3n+5}-370688y_{3n+4}+1026x_{n+2}-32736y_{n+1}\right]$
- $\frac{1}{49}\left[393174x_{3n+5}-370688y_{3n+5}+1026x_{n+1}-960y_{n+1}\right]$

- $\frac{1}{882}\left[342y_{3n+4}-11574y_{3n+3}+1026y_{n+2}-34722y_{n+1}\right]$
- $\frac{1}{29988}\left[342y_{3n+5}-393174y_{3n+3}+1026y_{n+3}-1179522y_{n+1}\right]$
- $\frac{1}{882}\left[11574y_{3n+5}-393174y_{3n+4}+1026y_{n+2}-34722y_{n+1}\right]$

5. Each of the following is a bi quadratic integer

- $\frac{1}{784}\left[10912x_{4n+4}-320x_{4n+5}+43648x_{2n+2}-1280x_{2n+3}+4704\right]$
- $\frac{1}{26656}\left[370688x_{4n+4}-320x_{4n+6}+1482752x_{2n+2}\_1280x_{2n+4}+159936\right]$
- $\frac{1}{49}\left[342x_{4n+4}-320y_{4n+4}+1368x_{2n+2}-1280y_{2n+2}+294\right]$
- $\frac{1}{833}\left[11574x_{4n+4}-320y_{4n+5}+46296x_{2n+2}-1280y_{2n+3}+4998\right]$
- $\frac{1}{28273}\left[393174x_{4n+4}-320y_{4n+6}+1572696x_{2n+2}-1280y_{2n+4}+169638\right]$
- $\frac{1}{784}\left[370688x_{4n+5}-10912x_{4n+6}+1482752x_{2n+3}-43648x_{2n+4}+4704\right]$
- $\frac{1}{833}\left[342x_{4n+5}-10912y_{4n+4}+1368x_{2n+3}-43648y_{2n+2}+4998\right]$
- $\frac{1}{49}\left[11574x_{4n+4}-10912y_{4n+5}+46296x_{2n+2}-43648y_{2n+3}+294\right]$
- $\frac{1}{833}\left[393174x_{4n+5}-10912y_{4n+5}-1572696x_{2n+3}-43648y_{2n+3}+4998\right]$

- $\frac{1}{28273}\left[343x_{4n+6} - 370688y_{4n+4} + 1372x_{2n+4} - 1482752x_{2n+2} + 169638\right]$
- $\frac{1}{833}\left[11574x_{4n+6} - 370688y_{4n+5} + 46296x_{2n+4} - 1482752y_{2n+3} + 4998\right]$
- $\frac{1}{49}\left[393174x_{4n+6} - 370688y_{4n+6} + 1572696x_{2n+4} - 1482752y_{2n+4} + 294\right]$
- $\frac{1}{882}\left[342y_{4n+5} - 11574y_{4n+4} + 1368y_{2n+3} - 46296y_{2n+2} + 5292\right]$
- $\frac{1}{29988}\left[342y_{4n+6} - 393174y_{4n+4} + 1368y_{2n+4} - 1572696y_{2n+2} + 179928\right]$
- $\frac{1}{882}\left[11574y_{4n+6} - 393174y_{4n+5} + 46296y_{2n+4} - 1572696y_{2n+3} + 5292\right]$

6. Each of the following is a quintic integer

- $\frac{1}{784}\left[10912x_{5n+5} - 320x_{5n+6} + 54560x_{3n+3} - 1600x_{3n+4} - 109120x_{n+1} + 3200x_{n+2}\right]$
- $\frac{1}{26656}\left[370688x_{5n+5} - 320x_{5n+7} + 1853440x_{3n+3} - 4800x_{3n+5} - 3706880x_{n+1} + 3200x_{n+3}\right]$
- $\frac{1}{49}\left[342x_{5n+5} - 320y_{5n+5} + 1710x_{3n+3} - 1600y_{3n+3} - 3420x_{n+1} + 3200y_{n+1}\right]$
- $\frac{1}{833}\left[11574x_{5n+5} - 320y_{5n+7} + 57870x_{3n+3} - 1600y_{3n+5} - 115740x_{n+1} + 3200y_{n+3}\right]$
- $\frac{1}{784}\left[370688x_{5n+6} - 10912x_{5n+7} + 1853440x_{3n+4} - 54560x_{3n+5} - 3706880x_{n+2} + 109120x_{n+3}\right]$
- $\frac{1}{833}\left[342x_{5n+6} - 10912y_{5n+5} + 1710x_{3n+4} - 54560y_{3n+3} - 3420x_{n+2} + 10912y_{n+1}\right]$
- $\frac{1}{49}\left[11574x_{5n+6} - 10912y_{5n+6} + 57870x_{3n+4} - 54560y_{3n+4} - 115740x_{n+2} + 109120y_{n+2}\right]$
- $\frac{1}{28273}\left[343x_{5n+7} - 370688y_{5n+5} + 1715x_{3n+5} - 1853440y_{3n+3} - 3430x_{n+3} + 3706880y_{n+1}\right]$

- $\frac{1}{833}\left[11574x_{5n+7}-370688y_{5n+6}+57870x_{3n+5}-1853440y_{3n+4}-115740x_{n+3}+3706880y_{n+2}\right]$
- $\frac{1}{49}\left[393174x_{5n+7}-370688y_{5n+7}+1965870x_{3n+5}-1853440y_{3n+5}-3931740x_{n+3}+3706880y_{n+3}\right]$
- $\frac{1}{882}\left[342y_{5n+6}-11574y_{5n+5}+1710y_{3n+4}-57870y_{3n+3}-3420y_{n+2}+115740y_{n+1}\right]$
- $\frac{1}{29988}\left[342y_{5n+7}-393174y_{5n+5}+1710y_{3n+5}-1965870y_{3n+3}-3420y_{n+3}+3931740y_{n+1}\right]$
- $\frac{1}{28273}\left[393174x_{5n+5}-320y_{5n+7}+1965870x_{3n+3}-1600y_{3n+5}-3931740x_{n+1}+3200y_{n+3}\right]$
- $\frac{1}{833}\left[393174x_{5n+6}-10912y_{5n+6}+1965870x_{3n+4}-54560y_{3n+4}-3931740x_{n+2}+109120y_{n+2}\right]$
- $\frac{1}{882}\left[11574y_{5n+7}-393174y_{5n+6}+57870y_{3n+5}-1965870y_{3n+4}-115740y_{n+3}+3931740y_{n+2}\right]$

7. Relations among the solutions are given below:

- $16y_{n+1}+17x_{n+1}-x_{n+2}=0$
- $16y_{n+2}+x_{n+1}-17x_{n+2}=0$
- $x_{n+1}+x_{n+3}-34x_{n+2}=0$
- $544y_{n+1}+577x_{n+1}-x_{n+3}=0$
- $32y_{n+2}+x_{n+1}-x_{n+3}=0$
- $544y_{n+3}+x_{n+1}-577x_{n+3}=0$
- $18x_{n+1}+17y_{n+1}-y_{n+2}=0$
- $612x_{n+1}+577y_{n+1}-y_{n+3}=0$
- $16y_{n+2}+x_{n+1}-17x_{n+2}=0$
- $18x_{n+1}+577y_{n+2}-17y_{n+3}=0$
- $y_{n+1}+18x_{n+2}-17y_{n+2}=0$

- $17x_{n+1}+16y_{n+3}-577x_{n+2}=0$
- $577x_{n+2}+16y_{n+1}-17x_{n+3}=0$
- $18x_{n+3}+17y_{n+1}-577y_{n+2}=0$
- $612x_{n+3}+y_{n+1}-577y_{n+3}=0$
- $17x_{n+2}+16y_{n+2}-x_{n+3}=0$
- $18x_{n+3}+y_{n+2}-17y_{n+3}=0$
- $x_{n+2}+16y_{n+3}-17x_{n+3}=0$
- $36x_{n+2}+y_{n+1}-y_{n+3}=0$
- $y_{n+1}+y_{n+3}-34y_{n+2}=0$
- $18x_{n+2}+17y_{n+2}-y_{n+3}=0$
- $18x_{n+3}+y_{n+2}-17y_{n+3}=0$

3. REMARKABLE OBSERVATIONS:

3.1 Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the table 2 below.

Table 2:Illustrations

S.no	Hyperbola	(X, Y)
1	$X_n^2-72Y_n^2=2458624$	$(10912x_{n+1}-320x_{n+2}, 38x_{n+2}-1286x_{n+1})$
2	$X_n^2-72Y_n^2=2842169344$	$(370688x_{n+1}-320x_{n+3}, 38x_{n+3}-43686x_{n+1})$
3	$X_n^2-72Y_n^2=9604$	$(342x_{n+1}-320y_{n+1}, 38y_{n+1}-40x_{n+1})$
4	$X_n^2-72Y_n^2=2775556$	$(11574x_{n+1}-320y_{n+2}, 38y_{n+2}-1364x_{n+1})$
5	$X_n^2-72Y_n^2=3197450116$	$(393174x_{n+1}-320y_{n+3}, 38y_{n+3}-46336x_{n+1})$
6	$X_n^2-72Y_n^2=2458624$	$(370688x_{n+2}-10912x_{n+3}, 1286x_{n+3}-43686x_{n+2})$

7	$X_n^2 - 72Y_n^2 = 27755556$	$(342x_{n+2} - 10912y_{n+1}, 1286y_{n+1} - 40x_{n+2})$
8	$X_n^2 - 72Y_n^2 = 9604$	$(11574x_{n+2} - 10912y_{n+2}, 1286y_{n+2} - 1364x_{n+2})$
9	$X_n^2 - 72Y_n^2 = 2775556$	$(393174\ x_{n+2} - 10912\ y_{n+2}, 1286\ y_{n+3} - 46336\ x_{n+2})$
10	$X_n^2 - 72Y_n^2 = 3197450116$	$(343x_{n+3} - 370688\ y_{n+1}, 43686\ y_{n+1} - 40x_{n+3})$
11	$X_n^2 - 72Y_n^2 = 2775556$	$(11574x_{n+3} - 370688y_{n+2}, 43686y_{n+2} - 1364x_{n+3})$
12	$X_n^2 - 72Y_n^2 = 9604$	$(393174\ x_{n+3} - 370688\ y_{n+3}, 43686\ y_{n+3} - 46336\ x_{n+3})$
13	$X_n^2 - 72Y_n^2 = 3111696$	$(342\ y_{n+2} - 11574\ y_{n+1}, 1364\ y_{n+1} - 40\ y_{n+2})$
14	$X_n^2 - 72Y_n^2 = 3597120576$	$(342y_{n+3} - 393174y_{n+1}, 46336y_{n+1} - 40y_{n+3})$
15	$X_n^2 - 72Y_n^2 = 3111696$	$(11574y_{n+3} - 393174y_{n+2}, 46336y_{n+2} - 1364y_{n+3})$

3.2 Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table 3 below

Table 3:Illustrations

S.no	**Parabola**	(X, Y)
1	$784X_n - 72Y_n^2 = 1229312$	$(370688x_{2n+3} - 10912x_{2n+4}, 1286x_{n+3} - 43686x_{n+2})$
2	$26656X_n - Y_n^2 = 1421084672$	$(370688x_{2n+2} - 320x_{2n+4}, 38x_{n+3} - 43686x_{n+1})$
3	$49X_n - 72Y_n^2 = 4802$	$(342x_{2n+2} - 320y_{2n+2}, 38y_{n+1} - 40x_{n+1})$
4	$833X_n - 72Y_n^2 = 1387778$	$(11574x_{2n+2} - 320y_{2n+3}, 38y_{n+2} - 1364x_{n+1})$
5	$28273X_n - 72Y_n^2 = 1598725058$	$(393174x_{2n+2} - 320y_{2n+4}, 38y_{n+3} - 46336x_{n+1})$

6	$784X_n - 72Y_n^2 = 1229312$	$(370688x_{2n+3} - 10912x_{2n+4}, 1286x_{n+3} - 43686x_{n+2})$
7	$833X_n - 72Y_n^2 = 1387778$	$(342x_{2n+3} - 10912y_{2n+2}, 1286y_{n+1} - 40x_{n+2})$
8	$49X_n - 72Y_n^2 = 4802$	$(11574x_{2n+3} - 10912y_{2n+3}, 1286y_{n+2} - 1364x_{n+2})$
9	$833X_n - 72Y_n^2 = 1387778$	$(11574x_{2n+4} - 370688y_{2n+3}, 43686y_{n+2} - 1364x_{n+3})$
10	$28273X_n - 72Y_n^2 = 1598725058$	$(343x_{2n+4} - 370688y_{2n+2}, 43686y_{n+1} - 40x_{n+3})$
11	$833X_n - 72Y_n^2 = 1387778$	$(393174x_{2n+3} - 10912y_{2n+3}, 1286y_{n+3} - 46336x_{n+2})$
12	$49X_n - 72Y_n^2 = 4802$	$(393174x_{2n+4} - 370688y_{2n+4}, 43686y_{n+3} - 43686x_{n+3})$
13	$882X_n - 72Y_n^2 = 1555848$	$(209x_{n+3} - 1257y_{n+2}, 4169y_{2n+3} - 693x_{2n+4})$
14	$29988X_n - 72Y_n^2 = 1798560288$	$(342y_{2n+4} - 393174y_{2n+2}, 46336y_{n+1} - 40y_{n+3})$
15	$882X_n - 72Y_n^2 = 1555848$	$(11574y_{2n+4} - 393174y_{2n+3}, 46336y_{n+2} - 1364y_{n+3})$

3.3 Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of straight lines which are presented in the table 4 below

Table 4:Illustrations

S.no	**Straight line**	**(X, Y)**
1.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 11584x_{n+1} - 10x_{n+3}$

2.	$X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 342x_{n+1} - 320y_{n+1}$
3.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 11584x_{n+1} - 320y_{n+2}$
4.	$X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 23268x_{n+2} - 682x_{n+3}$
5.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 342x_{n+2} - 10912y_{n+1}$
6.	$X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 11574x_{n+2} - 10912y_{n+2}$
7.	$577X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 342x_{n+3} - 370688y_{n+1}$
8.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 11574x_{n+3} - 370688y_{n+2}$
9.	$9X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 171y_{n+2} - 5787y_{n+1}$
10.	$577X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 393174x_{n+1} - 320y_{n+3}$
11.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 393174x_{n+2} - 10912y_{n+3}$

12.	$306X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 171y_{n+3} - 196587y_{n+1}$
13.	$9X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 5787y_{n+3} - 196587y_{n+2}$
14.	$X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 393174x_{n+3} - 370688y_{n+3}$
15.	$X = 17Y$	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 342x_{n+1} - 320y_{n+1}$
16.	$X = Y$	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 11584x_{n+1} - 320y_{n+2}$
17.	$X = 17Y$	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 23268x_{n+2} - 682x_{n+3}$
18.	$X = Y$	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 342x_{n+2} - 10912y_{n+1}$
19.	$X = 17Y$	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 11574x_{n+2} - 10912y_{n+2}$
20.	$577X = Y$	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 342x_{n+3} - 370688y_{n+1}$
21.	$17X = Y$	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 11574x_{n+3} - 370688y_{n+2}$

22.	9X = Y	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 171y_{n+2} - 5787y_{n+1}$
23.	577X = Y	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 393174x_{n+2} - 320y_{n+3}$
24.	17X = Y	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 393174x_{n+2} - 10912y_{n+3}$
25.	306X = Y	$X = 11584x_{n+1} - 10x_{n+3}$ $Y = 171y_{n+3} - 196587y_{n+1}$

3.4 Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}$ observe that $p > q > 0$ .Treat p, q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, where $\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2$. Let A, P represent the area and perimeter of $T(\alpha, \beta, \gamma)$. Then the following interesting relations are observed.

a) $16X - 9Y - 7Z - 98 = 0$.

b) $\frac{2A}{P} = x_{n+1} y_{n+1}$.

c) $3(Z - Y)$ is a nasty number.

d) $3(X - \frac{4A}{P})$ is a nasty number.

e) $X - \frac{4A}{P} + Y$ is written as the sum of two squares.

3.5 From the values of y_{n+1}, one may generate second order Ramanujan numbers with base numbers as real integers and Gaussian integers.

Illustration : Consider $y_0 = 20 = 20 * 1 = 2 * 10 = 4 * 5$

Now, $20*1=2*10 \Rightarrow (20+1)^2+(10-2)^2=(20-1)^2+(10+2)^2$

$$\Rightarrow 21^2+8^2=19^2+12^2=505$$

In a similar manner,

$$2*10=4*5 \Rightarrow 12^2+1^2=8^2+9^2=145$$
$$20*1=4*5 \Rightarrow 21^2+1^2=19^2+9^2=442$$

Thus 505, 145, 442 are second order Ramanujan numbers with base numbers as real integers.

Also,

$$20*1=2*10 \Rightarrow (20+i)^2+(10-2i)^2=(20-i)^2+(10+2i)^2=495$$
$$2*10=4*5 \Rightarrow (10+2i)^2+(5-4i)^2=(10-2i)^2+(5+4i)^2=105$$
$$20*1=4*5 \Rightarrow (20-i)^2+(5+4i)^2=(20+i)^2+(5-4i)^2=408$$

Here 495, 105, 408 are second order Ramanujan numbers with base numbers as Gaussian integers.

3.6 Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where $a_{n+1}=\frac{x_{n+1}-1}{2}, b_{n+1}=\frac{y_{n+1}}{2}$

It is observed that

a) $6(t_{3,a_{n+1}}-5)$ is a Nasty Number.

b) $9t_{3,a_{n+1}}-t_{10,b_{n+1}} \equiv 2 \pmod 3$

c) $9t_{3,a_{n+1}}-4b_{n+1}^2=5$ in which $t_{m,n}$ represents a polygonal number of rank n with side m.

REMARK 1:

One may also employ the linear transformations $x=X-8T$, $y=X-9T$ to solve (1) and obtain a different set of solutions.

REMARK 2:

The introduction of the linear transformations $x=7(X\pm 8T)$, $y=7(X\pm 9T)$ in (1) leads to the pellian equation

$$X^2=72T^2+1$$

whose solutions are well –known. Applying these values in the above transformations, yet another set of integer solutions to (1) is obtained.

In this paper, a study is made for determining many integer solutions to the hyperbola represented by the Pell-Like equation $9x^2 - 8y^2 = 49$. As the quadratic equations are rich in variety, the readers of this paper may attempt to obtain integer solutions to other choices of quadratic equations with two or more unknowns.

REFERENCES

4.1. Journal Article

[1] *Gopalan,M.A., and Parvathy,G., (2010) "Integral Points On The Hyperbola* $x^2 + 4xy + y^2 - 2x - 10y + 24 = 0$ *,"Antarctica J.Math,Vol 1(2),149-155.*

[2] *Gopalan,M.A.,Gokila,K.,and Vidhyalakahmi,S., (2007) "On the Diophantine Equation* $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$ *", Acta Ciencia Indica, Vol.XXXIIIM No2, p. 567-570.*

[3] *Gopalan,M.A.,Vidhyalakahmi,S.,and Devibala,S., (2007), On The Diophantine Equation* $3x^2 + xy = 14$ *,Acta Ciencia Indica,Vol.XXXIII M.No2,P.645-646.*

[4] *Gopalan.M.A., and Janaki.G., (2008) "Observations on* $x^2 - y^2 + x + y + xy = 2$ *", Impact J.Sci.,Tech, Vol2(3)p.14, 3-148.*

[5] *Gopalan.M.A., and., Shanmuganadham,P.,and Vijayashankar,A., (2008) "On Binary Quadratic Equation* $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$ *", Acta Ciencia Indica,Vol . XXXIVM. No.4, p.1803-1805.*

[6] *Mollion,R.A, (1998) "All Solutions of the Diophatine Equations* $X^2 - DY^2 = n$ *", Far East J,Math.Sci., Speical Volume, Part III,p.257-293.*

[7] *Gopalan.M.A, Vidhyalakahmi.S and Premalatha.E , Observations On the Binary Quadratic Diophantine Equation* $y^2 = 8x^2 + 8x + 16$ *, International Journal Of Latest Research in Science and Technology,Vol.1,No(4), Pp 379-382, Nov-Dec 2012.*

[8] *Gopalan.M.A, Vidhyalakahmi.S and Premalatha.E , On the non-homogeneous binary quadratic equation* $x^2 - 3xy + y^2 + 2x = 0$ *, International journal of Engineering & Scientific Research, Vol 2, issue 2, Pp 83-88, Feb 2014.*

[9] *Gopalan.M.A, Vidhyalakahmi.S, Premalatha.E and Vanitha.M , Observation on the hyperbola* $y^2 = 220x^2 + 9$ *, Star Research Journal, Vol.4, issue 3(2), Pp 6-10, March 2016.*

[10] *Gopalan.M.A, Vidhyalakahmi.S, Premalatha.E and Sofia Christinal.S, On the negative pell equation* $y^2 = 72x^2 - 8$*, International Journal of Emerging Technologies in Engineering Research, Vol 4,issue 2, Pp 25-28, Feb 2016.*

[11] *Gopalan.M.A, Vidhyalakahmi.S, Premalatha.E and Janani.R, On the negative pell equation* $y^2 = 33x^2 - 8$ *, International Journal of Multidisciplinary Research and Modern Education, Vol 2,issue 1, Pp 95-100, 2016.*

[12] *Gopalan.M.A, N. Anusheela and Shreemathi Adiga, Observation on the negative pell equation* $y^2 = 90x^2 - 36$ *, International Journal of Latest Research in Science and Technology , Vol 7,issue 3, Pp 24-26, 2018.*

[13] *Gopalan.M.A and Sharadha Kumar , On the Pell- Like equation* $3x^2 - 8y^2 = 40$*, IJMR , Vol 5, issue 1, Pp 95-106, 2019.*

[14] *Vidhyalakahmi.S, Premalatha.E and Maheshwari . D, On the non homogeneous binary quadratic equation* $4x^2 - 3y^2 = 37$*, Bulletin of Pure and Applied Sciences- Mathematics and Statistics , Vol 38E,issue 1, Pp 324-328, 2019.*

4.2. Book

[15] *Banumurthy.T.S., (1995)A Modern Introduction to Ancient Indian Mathematics,Wiley Eastern Limited, London.*

[16] *Carmichael, R.D., (1950)The Theory of Numbers and Diophantine Analysis, Dover Publications, New York.*

[17] *Dickson. L. E., (1952)History of The Theory of Numbers, Vol.II, Chelsia Publicating Co, New York.*

[18] *Mordell,L,J., (1969)Diophantine Equations, Acadamic Press,London.*

[19] *Nigel,P.Smart., (1999)TheAlgorithm Resolutions of Diaphantine eqations,Cambridge University, Press, London.*

[20] *Telang ,S.G., (1996) Number theory, Tata Mc Graw-Hill Publishing Company , NewDelhi.s*