Fabrication of Gorgeous Integer Quadruple

V. Pandichelvi[1]

P. Sandhya[2]

[1]*Assistant Professor, PG & Research Department of Mathematics*

Urumu Dhanalakshmi College, Trichy, India.

[2]*Assistant Professor, PG & Research Department of Mathematics*

SRM Trichy Arts & Science College, Trichy, India.

Abstract: In this paper, an elegant non-zero distinct integer quadruple (a, b, c, d) in which addition of any three of them is a cubical integer is determined by exploiting the general solutions to a meticulous cubic Diophantine equation.

Keywords: Diophantine triples, Ternary quadratic Diophantine equation.

1.Introduction

Diophantus of Alexandria noted that the numbers $\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}$ had the property that the product of either of these two numbers increased by 1 is the square of a rational number. Sets of integers with a comparable property have been of concern for many years, and a sequence of non- negative integers, is verbalized to be a Diophantine m-tuple $\{a_1, a_2, \dots, a_m\}$ with property $D(n)$ if each $a_i a_j + n (i \neq j)$ is the square of an integer [1-7.10]. A variety of integer solutions to different Diophantine equations are analysed in [8,9]. In this communication, the Diophantine quadruple consisting of non-zero distinct integers where the sum of any three elements is a cubic of an integer is discovered.

2. Method of Analysis

Let a, b, c, d be four non-zero distinct integers such that the addition of any three of them is a perfect cube.

Consider

$$a + b + c = p^3 \quad (1)$$

$$a + b + d = q^3 \quad (2)$$

$$a + c + d = r^3 \tag{3}$$

$$b + c + d = s^3 \tag{4}$$

together with the following condition

$$3(a + b + c + d) = (p + q + r + s)z^3 \tag{5}$$

Solving the system of equations from (1) to (4), the corresponding values of a, b, c, d are pointed out by

$$a = \frac{1}{3}(p^3 + q^3 + r^3 - 2s^3) \tag{6}$$

$$b = \frac{1}{3}(p^3 + q^3 + s^3 - 2r^3) \tag{7}$$

$$c = \frac{1}{3}(p^3 + r^3 + s^3 - 2q^3) \tag{8}$$

$$d = \frac{1}{3}(q^3 + r^3 + s^3 - 2p^3) \tag{9}$$

Adding (6), (7), (8) and (9), an interesting combination is enumerated by

$$3(a + b + c + d) = p^3 + q^3 + r^3 + s^3 \tag{10}$$

Comparison of (5) & (10) provides that

$$(p + q + r + s)z^3 = p^3 + q^3 + r^3 + s^3 \tag{11}$$

Employing the following linear transformations

$p = x + 2y, q = 2x + y, r = 2y - x, s = y - 2x$, where x and y are non-zero integers

from (6) to (9) gives

$$a = 8x^3 + 6xy^2 + 5y^3 \tag{12}$$

$$b = x^3 + 6x^2y + 12xy^2 - 2y^3 \tag{13}$$

$$c = -8x^3 - 6xy^2 + 5y^3 \tag{14}$$

$$d = -x^3 + 6x^2y - 12xy^2 - 2y^3 \tag{15}$$

Substitution of the same transformations reduce (11) to the quadratic equation with three unknowns as

$$6x^2 + 3y^2 = z^3 \tag{16}$$

Applying four different procedures of solving (16), the evaluation of an attractive integer quadruple satisfying the condition that the sum of any three quantities is a cubical integer is explained as follows.

Procedure (i):

The choice of $z = 6m^2 + 3n^2$, where $m, n \in Z - \{0\}$, leads (16) to

$$\left(\sqrt{6}x\right)^2+\left(\sqrt{3}y\right)^2=\left(\left(\sqrt{6}m\right)^2+\left(\sqrt{3}n\right)^2\right)^3$$

which implies that

$$\left(\sqrt{6}x+i\sqrt{3}y\right)\left(\sqrt{6}x-i\sqrt{3}y\right)=\left(\left(\sqrt{6}m+i\sqrt{3}n\right)\left(\sqrt{6}m-i\sqrt{3}n\right)\right)^3$$

Escalating the right hand side of the above equation and equating real and imaginary parts on both the sides, it is to be noted that

$$x=6m^3-9mn^2$$

$$y=18m^2n-3n^3$$

$$z=6m^2+3n^2$$

Substituting the above values of x,y,z in (12), (13), (14) and (15), the values of a,b,c,d satisfying our assumption are deliberated by

$$a=8(6m^3-9mn^2)^3+6(18m^2n-3n^3)^2(6m^3-9mn^2)$$
$$+5(18m^2n-3n^3)^3$$

$$b=(6m^3-9mn^2)^3+6(6m^3-9mn^2)^2(18m^2n-3n^3)$$
$$+12(6m^3-9mn^2)\times(18m^2n-3n^3)^2-2(18m^2n-3n^3)^3$$

$$c=-8(6m^3-9mn^2)^3-6(6m^3-9mn^2)(18m^2n-3n^3)^2$$
$$+5(18m^2n-3n^3)^3$$

$$d=-(6m^3-9mn^2)^3+6(6m^3-9mn^2)^2(18m^2n-3n^3)$$
$$-12(6m^3-9mn^2)\times(18m^2n-3n^3)^2-2(18m^2n-3n^3)^3$$

Some numerical examples satisfying the hypothesis are specified below

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	12609	-14067	21141	2187	27^3	9^3	33^3	21^3
2	1	2715525	1456542	569565	-2025378	168^3	129^3	108^3	9^3
3	2	165419721	9726264	104580288	-107228664	654^3	408^3	546^3	192^3

Procedure (ii):

Treating (16) as

$$6x^2+3y^2=1^2.z^3 \quad (17)$$

Assuming that

$$z=\left(\sqrt{6}m\right)^2+\left(\sqrt{3}n\right)^2$$

and re-establish 1 by

$$1 = \frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}$$

in (17), it becomes

$$\left(\sqrt{6}x\right)^2 + \left(\sqrt{3}y\right)^2 = \left(\frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}\right)^2 . \left(\left(\sqrt{6}m\right)^2 + \left(\sqrt{3}n\right)^2\right)^3$$

which is equivalent to

$$\left(\sqrt{6}x + i\sqrt{3}y\right)\left(\sqrt{6}x - i\sqrt{3}y\right) = \left(\frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}\right)^2 . \left(\left(\sqrt{6}m + i\sqrt{3}n\right)\left(\sqrt{6}m - i\sqrt{3}n\right)\right)^3 \quad (18)$$

Equating the positive parts on both sides of (18) and comparing the like terms, it is examined that

$$x = 2m^3 - 12m^2n - 3mn^2 + 2n^3$$

$$y = 8m^3 + 6m^2n - 12mn^2 - n^3$$

In view of (12), (13), (14) and (15), the options of a, b, c, d are estimated by

$$a = 8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$b = (2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times (8m^3 + 6m^2n - 12mn^2 - n^3) + 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$c = -8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 - 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$d = -(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times (8m^3 + 6m^2n - 12mn^2 - n^3) - 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

Some numerical examples satisfying the propositions are specified below

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	-10709	-739	10719	2187	$(-9)^3$	$(-21)^3$	13^3	23^3
2	1	19683	-1771470	2480787	1751058	90^3	$(-9)^3$	162^3	135^3
3	2	-55002032	-46632952	105976512	94647096	162^3	$(-192)^3$	526^3	536^3

Procedure (iii):

Consider an alternative solutions to (16) as

$$x = \frac{1}{\sqrt{6}}(k^3 + kl^2)$$

$$y = \frac{1}{\sqrt{3}}(l^3 + lk^2)$$

$$z = k^2 + l^2$$

Case (i):

Since the target is to evaluate integral values for the variables, it is observed that the subsequent two parametric choices of $k = \sqrt{6}m$ and $l = \sqrt{3}n$ provides the values of x and y in integers.

Then, the integral solutions to (16) are calculated by

$$x = 6m^3 + 3mn^2$$

$$y = 3n^3 + 6nm^2$$

$$z = 6m^2 + 3n^2$$

Substituting the above quantities in (12), (13), (14) and (15), the appropriate values of a, b, c, d are discovered by

$$a = 1728m^9 + 3888m^7n^2 + 1080m^6n^3 + 3240m^5n^4 + 1620m^4n^5 + 1188m^3n^6 + 810m^2n^7 + 162mn^8 + 135n^9$$

$$b = 216m^9 + 1296m^8n + 2916m^7n^2 + 1512m^6n^3 + 4050m^5n^4 + 324m^4n^5 + 1971m^3n^6 - 162m^2n^7 + 324mn^8 - 54n^9$$

$$c = -1728m^9 - 3888m^7n^2 + 1080m^6n^3 - 3240m^5n^4 + 1620m^4n^5 - 1188m^3n^6 + 810m^2n^7 - 162mn^8 + 135n^9$$

$$d = -216m^9 + 1296m^8n - 2916m^7n^2 + 1512m^6n^3 - 4050m^5n^4 + 324m^4n^5 - 1971m^3n^6 - 162m^2n^7 - 324mn^8 - 54n^9$$

Some numerical examples satisfying our assumption are precised below

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	13851	12393	-6561	-6561	27^3	27^3	9^3	9^3
2	1	1594323	1062882	-1397493	-196830	108^3	135^3	0^3	$(-81)^3$
3	2	94298688	75611448	-71299008	-22712184	462^3	528^3	66^3	$(-264)^3$

Case (ii):

As in case (i), the single parametric choices of $k = \sqrt{6}t$ and $l = \sqrt{3}t$ offers the values of x and y in integers.

Thus,

$$x = 9t^3$$

$$y = 9t^3$$

$z = 9t^2$

Substituting the above magnitudes in (12), (13), (14) and (15), it is determined by

$a = 13851\ t^9$

$b = 12393\ t^9$

$c = -6561\ t^9$

$d = -6561\ t^9$

Some numerical examples satisfying the hypothesis are exemplified below

k	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	13851	12393	-6561	-6561	27^3	27^3	9^3	$(-9)^3$
2	7091712	6345216	-3359232	-3359232	216^3	216^3	72^3	$(-72)^3$
3	272629233	243931419	-129140163	-129140163	729^3	729^3	243^3	$(-243)^3$

The C Program for numerical examples satisfying our hypotheses are illustrated below.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
char ch;
intm,n,ca;
signed long int x,y,a,b,c,d,cup,cuq,cur,cus,cup1,cuq1,cur1,cus1,p,q,r,s;
clrscr();
do
{
printf("\nEnter m and n values\n");
scanf("%d%d",&m,&n);
printf("\nEnter your choice case 1 or 2 or 3 or 4\n");
scanf("%d",&ca);
switch(ca)
{
case 1:
x=(6*m*m*m)-(9*m*n*n);
y=(18*m*m*n)-(3*n*n*n);
break;
case 2:
x=(2*m*m*m)-(12*m*m*n)-(3*m*n*n)+(2*n*n*n);
y=(8*m*m*m)+(6*m*m*n)-(12*m*n*n)-(n*n*n);
break;
case 3:
x=(6*m*m*m)+(3*m*n*n);
y=(3*n*n*n)+(6*n*m*m);
break;
case 4:
x=9*m*m*m;
```

```
y=9*m*m*m;
break;
}
a=(8*x*x*x)+(6*x*y*y)+(5*y*y*y);
b=(x*x*x)+(6*x*x*y)+(12*x*y*y)-(2*y*y*y);
c=(-8*x*x*x)-(6*x*y*y)+(5*y*y*y);
d=-(x*x*x)+(6*x*x*y)-(12*x*y*y)-(2*y*y*y);
cup=a+b+c;
cuq=a+b+d;
cur=a+c+d;
cus=b+c+d;
if(cup<0)
{
cup1=-1*cup;
p=pow(cup1,1.0/3.0);
}
else
{
p=pow(cup,1.0/3.0);
}
If(p==0)
p=0;
else
p++;
if(cup<0)
p=-p;
if(cuq<0)
{
cuq1=-1*cuq;
q=pow(cuq1,1.0/3.0);
}
else
{
q=pow(cuq,1.0/3.0);
}
If(q==0)
q=0;
else
q++;
if(cuq<0)
q=-q;
if(cur<0)
{
cur1=-1*cur;
p=pow(cur1,1.0/3.0);
}
else
{
r=pow(cur,1.0/3.0);
}
```

```
If(r==0)
r=0;
else
r++;
if(cur<0)
r=-r;
if(cus<0)
{
cus1=-1*cus;
s=pow(cus1,1.0/3.0);
}
else
{
s=pow(cus,1.0/3.0);
}
if(s==0)
s=0;
else
s++;
if(cus<0)
s=-s;
printf("\nm=%d,n=%d,a=%ld,b=%ld,b=%ld,c=%ld,d=%ld\n",m,n,a,b,c,d);
printf("\na+b+c=(%ld)^3,a+b+d=(%ld)^3,a+c+d=(%ld)^3,b+c+d=(%ld)^3",p,q,r,s);
printf("Do you want to continue (y/n)?");
ch=getche();
}while (ch=='Y' || ch=='y');
getch();
}
```

3. Conclusion

In this communication, the quadruple (a,b,c,d) so that the sum of any three of them is a cubical integer is scrutinized. To conclude, one can search for different quintuples, sextuples, septuples etc satisfying some other properties.

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